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ON MEROMORPHICALLY MULTIVALENT FUNCTIONS

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Abstract. The purpose of this paper is to derive some properties of certain meromorphically multivalent functions in annulus.

1. Introduction

Let Σ_p be the class of functions of the form

(1.1) $$f(z) = 1/z^p + a_0/z^{p-1} + \cdots\cdots + a_{k+p-1}z^k + \cdots,$$

which are analytic in the annulus $D = \{ Z : |z| < 1 \}$, where $p \varepsilon N=\{1, 2, 3, \cdots\}$. For $f(z) \varepsilon \Sigma_p$, we define the operator $D^{n+p-1}f(z)$ by

(1.2) $$D^{n+p-1}f(z) = (z^{n+2p-1} f(z)/(n+p-1)!)^{(n+p-1)}/z^p$$

$$= 1/z^p + (n+p)a_0/z^{p-1} + (n+p)(n+p+1)a_1/(2!z^{p-2}) + \cdots$$

$$+(n+p)(n+p+1)\cdots(n+k+2p-1)a_{k+p-1}z^k/(k+p)! + \cdots,$$

where n is an integer and $n > -p$.

Recently, Cho and Nunokawa [1] proved that

$$\mathrm{Re}\{z^{p+1}(D^{n+p}f(z))'\} < -\alpha \quad (0 \leqq \alpha < p\ ;\ |z| < 1)$$

$$\text{implies}\quad \mathrm{Re}\{z^{p+1}(D^{n+p-1}f(z))'\} < -\beta \quad (|z| < 1)$$

where

$$\beta = (p + 2\alpha(n+p))/(1+2(n+p)).$$

In the present paper, we show another properties of functions $f(z) \varepsilon \Sigma_p$ concerning with the operator $D^{n+p-1} f(z)$.

2. Main results

We need the following lemma due to Jack [2] (or, due to Miller and Mocanu[3]).

Lemma. Let $w(z)$ be non-constant analytic in $U = \{ Z: |z| < 1 \}$ with $w(0)=1$. If $|w(z)|$ attains its maximum value at a point z_0 on the circle $|z|=r<1$, then we have

$$z_0 w'(z_0) = kw(z_0)$$

where k is real and $k \geqq 1$.

Theorem 1. If $f(z) \varepsilon \Sigma_p$ satisfies

$$\text{(2.1)} \qquad \mathrm{Re}\{ z^{p+1} (D^{n+p} f(z))' \} > -\alpha \quad (z \varepsilon U)$$

for some α (α>p), then

$$\text{(2.2)} \qquad \mathrm{Re}\{ z^{p+1} (D^{n+p-1} f(z))' \} > -\beta \quad (z \varepsilon U),$$

where

$$\beta = (p +2\alpha(n+p))/(1 +2(n+p)).$$

Proof. Define the function $w(z)$ by

$$\text{(2.3)} \qquad z^{p+1} (D^{n+p-1} f(z))' = ((p -2\beta)w(z) -p)/(1+w(z)),$$

$w(z) \neq -1$, with

$$\beta = (p +2\alpha(n+p))/(1 +2(n+p)).$$

Then $w(z)$ is analytic in U and $w(0)=0$. Note that

$$\text{(2.4)} \qquad z(D^{n+p-1} f(z))' = (n+p)D^{n+p} f(z) -(n+2p) D^{n+p-1} f(z).$$

It follows from (2.3) that

$$\text{(2.5)} \qquad (n+p) z^p D^{n+p} f(z) -(n+2p) z^p D^{n+p-1} f(z)$$

$$= ((p-2\beta)w(z)-p)/(1+w(z)).$$

Taking the differentiations in both sides of (2.5), we have

$$(2.6)\qquad z^{p+1}(D^{n+p}f(z))' = ((p-2\beta)w(z)-p)/(1+w(z))$$

$$+\,2(p-\beta)zw'(z)/((n+p)(1+w(z))^2).$$

Suppose that there exists a point $z_0 \varepsilon U$ such that

$$\max_{|z|\leqq|z_0|}|w(z)| = |w(z_0)| \qquad (w(z_0) \neq -1),$$

then, by Lemma, we have

$$z_0 w'(z_0) = kw(z_0) \qquad (k \geqq 1).$$

Therefore, letting $w(z_0) = e^{i\theta}$ $(0 \leqq \theta \leqq 2\pi)$, we see that

$$(2.7)\qquad \mathrm{Re}\{z_0^{p+1}(D^{n+p}f(z_0))'\} + \alpha$$

$$= \alpha + \mathrm{Re}\{((p-2\beta)e^{i\theta}-p)/(1+e^{i\theta})\} + 2(p-\beta)k(n+p)^{-1}\mathrm{Re}\{e^{i\theta}(1+e^{i\theta})^{-2}\}$$

$$= \alpha - \beta + (p-\beta)k/((n+p)(1+\cos\theta))$$

$$\leqq \alpha - \beta + (p-\beta)/2(n+p) = 0$$

for $\alpha > p$ and $\beta = (p+2\alpha(n+p))/(1+2(n+p))$.

This contradicts our condition (2.1). Therefore, $|w(z)| < 1$ for all $z \varepsilon U$, or

$$\mathrm{Re}\{z^{p+1}(D^{n+p-1}f(z))'\} > -\beta \qquad (z \varepsilon U).$$

Next, we prove

Theorem 2. Let

$$(2.8)\qquad F_c(z) = cz^{-c-p}\int_0^z t^{c+p-1}f(t)dt \qquad (c > 0)$$

for $f(z) \varepsilon \Sigma_p$. If $f(z)$ satisfies

(2.9) $$\mathrm{Re}\{z^{p+1}(D^{n+p-1}f(z))'\} > -\alpha \qquad (z \varepsilon U)$$

for some α ($\alpha > p$), then

(2.10) $$\mathrm{Re}\{z^{p+1}(D^{n+p-1}F_c(z))'\} > -\beta, \qquad (z \varepsilon u)$$

where $\beta = (p+2\alpha c)/(1+2c)$.

Proof. We define the function $w(z)$ by

(2.11) $$z^{p+1}(D^{n+p-1}F_c(z))' = ((p-2\beta)w(z)-p)/(1+w(z)) \qquad (w(z) \neq -1).$$

Then $w(z)$ is analytic in U and $w(0)=0$. Noting that

(2.12) $$z(D^{n+p-1}F_c(z))' = cD^{n+p-1}f(z) - (c+p)D^{n+p-1}F_c(z),$$

therefore we have

(2.13) $$z^{p+1}(D^{n+p-1}f(z))'$$

$$= ((p-2\beta)w(z)-p)/(1+w(z)) + 2(p-\beta)zw'(z)/c(1+w(z))^2).$$

Therefore, if we assume that there exists a point $z_0 \varepsilon U$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |W(Z_0)| = 1 \qquad (w(z_0) \neq -1)$$

then Lemma gives us that

(2.14) $$\mathrm{Re}\{z_0^{p+1}(D^{n+p-1}f(z_0))'\} + \alpha$$

$$\leq \alpha - \beta + (p-\beta)/2c$$

$$= 0$$

which contradicts our condition (2.9). This completes the proof of Theorem 2.

References

[1] N.E.Cho and M.Nunokawa, On certain subclass of meromorphically multivalent functions, Chinese J. Math. 22(1994), 197-202.

[2] I.S.Jack, Functions starlike and convex of order α, J. London Math. Soc. 3(1971), 469-474.

[3] S.S.Miller and P.T.Mocanu, Second order differential inequalities in the complex plane, J. Math. Anal. Appl. 65(1978), 289-305.

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